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Optimal dynamic investment policies under concave–convex adjustment costs*

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The paper considers a dynamic investment model of a monopolistic firm facing adjustment costs that are concave–convex in the rate of gross investment. The firm wishes to maximize the discounted stream of dividends over a finite planning horizon, plus the terminal value of the stock of equity. The problem of finding an optimal investment path is solved by combining results from a (nonstandard) model with concave adjustment costs with results from the (standard) model with convex adjustment costs. The results are presented in phase diagrams and are economically interpreted. The solution involves the use of a chattering control, and we discuss various approaches to avoid the occurrence of such a control policy.

1. Introduction

In early models of optimal dynamic investment [for instance, Jorgenson (1963)] the firm could adjust its capital stock *instantaneously* and *costlessly*. The former assumption can be relaxed by introducing a time lag reflecting delivery and installation delays or by imposing financial constraints. The latter assumption can be relaxed by incorporating adjustment costs. In this paper we study the influence of financial limits and adjustment costs on the optimal investment behaviour of the firm.

Adjustment costs derive their rationale from the observation that there are specific costs associated with the installation of capital goods. Intuitively one could think of adjustment costs as those costs incurred by the sale, purchase,

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and installation of capital goods, over and above the normal prices of these goods.

The influence of adjustment costs on the firm's optimal investment behaviour has been studied in a series of contributions. For a survey, see Söderström (1976).

The main research questions related to adjustment costs seem to be the following:

* What are the sources of adjustment costs?
* How should they be modelled?
* What are the implications of adjustment costs on the firm's optimal investment policy?

It is customary to make a distinction between firm-specific, *internal* adjustment costs (arising, for instance, by temporary decreases in productivity caused by the installation of new machinery) and *external* adjustment costs (typically, rising prices of new equipment in a monopsonistic capital goods market). Internal adjustment costs have been modelled by introducing a generalized production function, taking output as a function of the stocks of capital and labour as well as the investment rate: $Q = Q(K, L, I)$. A convenient formulation of the production function is the separable one where $Q = F(K, L) - C(I)$, where function C could be termed the adjustment cost function. In what follows we shall not be concerned with the distinction between internal and external costs, partly because the distinction is not unambiguous, partly because both types of costs may lead to the same formal model [cf. Lucas (1967), Gould (1968)].

As to modelling, two questions arise: what should be the *argument(s)* of the function and what *shape* could be supposed for the adjustment cost function, C?

Eisner and Strotz (1963), Mortensen (1973), and Brechling (1975) choose *net* investment as the argument of C. This choice is probably motivated by the solution technique employed, viz., the calculus of variations. Gould (1968) discusses the problem but settles for *gross* investment as the argument of C. This choice is also made by Lucas (1967), Rothschild (1971), Nickell (1978), Davidson and Harris (1981), El-Hodiri and Takayama (1981), Dechert (1984), and Kort (1989). Technically, the implication of adjustment costs depending on gross investment is that the firm will remain of bounded size [even in case of constant returns to scale in production; cf. Koskenkylä (1985)].

We conjecture that the choice of gross or net investment as argument of the adjustment cost function is not significant as long as the capital stock depreciates exponentially [cf. Treadway (1970)], supposing that C is a total cost, comprised of internal and external components. See also Gould (1968). [In a vintage capital model (i.e., abandoning the exponential decay assumption), Nickell (1975) employs a cost function depending in a separable way on both gross and net investment.]

The problem of selecting the 'right' argument of the adjustment cost function does not seem to have been resolved yet and we shall not proceed the discussion any further.

Likewise, there does not seem to be general concensus on what should be the 'proper' shape of the adjustment cost function. We suppose that the shape of the adjustment cost function is likely to produce more dramatic changes in the firm's investment behaviour than those due to the choice of the argument of the function, at least as long as we assume exponential decay of the capital stock.

Four different shapes of the adjustment cost function have been suggested: linear, convex, concave, and concave–convex, reflecting constant, increasing, decreasing, and decreasing–increasing marginal costs of adjusting the capital stock [cf. Rothschild (1971), Söderström (1976)]. The implications on the firm's investment policy of each of these assumptions will be discussed in section 3.

The major part of the literature has assumed a *convex* adjustment cost function. The use of such a cost function can be realistic, for example, in a situation where a firm, operating in a monopsonistic environment, wishes to acquire capital. Then the firm faces increasing prices because of the increased demand for capital goods. Such dis-economies of scale associated with more rapid changes of the capital stock should induce the firm to adjust its capital stock more slowly ('Rome was not built in one day'). Strictly convex adjustment costs imply that it is more expensive to do adjustments quickly than slowly.

On the other hand, *concavity* can be defended as a reasonable hypothesis, in particular for 'low' rates of investment [Rothschild (1971), Nickell (1978)]. A concave cost function could be relevant when fixed ordering costs and quantity discounts are present, but could also arise due to indivisibilities, economies of information in training, and so forth [Rothschild (1971)]. However, if we consider internal adjustment costs this may be true for sufficiently large amounts of investment expenditure. But internal adjustment costs being convex for low investment rates (which could imply the installation of a new production line over a period of fifty years) makes little or no sense. When monopsonistic elements in the capital goods market are not predominant it may be reasonable to suppose that adjustment costs are concave for small rates of investment.

Global concavity may be too strong an assumption which leads us to consider the *concave–convex* adjustment cost function. Davidson and Harris (1981) argue that the adjustment cost function be concave–convex since there could be some initial economies to scale when adjusting the capital stock, but as the investment rate becomes larger, in a firm of a given size, average adjustment costs eventually increase. Put in another way: installing capital at a (too) high rate ultimately leads to increasing costs. [A concave–convex cost function has been studied by Eswaran et al. (1983) in a problem of optimal competitive extraction of exhaustible resources.]

The dynamic investment model to be studied in this paper is related to the one analyzed in Davidson and Harris (1981). In two respects our model differs from that of Davidson and Harris: we consider a *finite* planning horizon, include a *salvage value* term at the horizon date, and impose an *explicit upper bound* on the firm's investment rate. As will be demonstrated, these features imply a richer structure of the optimal investment path.

The basic idea of the theory considered in this paper is to describe the firm's investment decision problem, in fairly general terms and over the firm's entire lifetime. Fundamental components are the firm's objective, its production technology and financial structure. The principal aim of the theory is to suggest an optimal investment policy over the firm's planning period, combining elements of different approaches to normative investment theory: the investment/consumption models of macroeconomic growth theory and capital accumulation, the investment/depreciation model of Jorgensen, and its extensions, and – to a lesser degree – dynamic financial models.

Our theory (as its predecessors) prescribes an optimal investment policy over the firm's lifecycle. It remains to be seen whether our prescriptions are 'better' than those in the literature closely related to this paper (e.g., the Davidson and Harris paper). The settlement of this question would require a comparison and evaluation of the assumptions made: for such an assessment the reader is referred to the discussion above.

Section 2 presents the dynamic optimization problem of the firm and sections 3.1–3.3 contain the mathematical analysis of the optimal control problem as well as economic interpretations of the results. In section 3.4 we focus on the changes in the model set-up that have to be made to avoid the chattering investment policies that inevitably turn up in the concave–convex case. Section 4 concludes the paper.

2. A dynamic investment model

Consider a monopolistic firm seeking to maximize the owners' value of the firm, consisting of the discounted stream of dividends over the planning period plus the discounted value of the amount of equity at the end of the planning period, that is,

$$\int_0^T \exp(-rt) D(t)\,\mathrm{d}t + \exp(-rT) X(T), \tag{1}$$

where $D = D(t)$ is the rate of dividend payout at time t, $X = X(t)$ the stock of equity by time t, $r = \text{constant} \geq 0$ the shareholders' time preference rate, and T the length of the firm's planning period which is fixed and finite.

Let $I = I(t)$ be the rate of gross investment at time t, $K = K(t)$ the stock of capital by time t, and $a = \text{constant} > 0$ the depreciation rate. The equation for the evolution of the stock of capital becomes

$$\dot{K} = I - aK, \qquad K(0) = K_0 = \text{constant} > 0. \tag{2}$$

We assume that the firm's only asset is its stock of capital goods which, in turn, can only be financed by equity (retained earnings). The firm is not able to borrow funds or issue new shares.

Remark 1. Empirically [cf. Sinn (1987)] issues of new shares have turned out to be a marginal means of finance in postwar Western economies. The assumption of no borrowing is more restrictive since in practice debt is an important means of finance. Admittedly, the reason for this assumption is mathematical convenience but we suspect that the inclusion of borrowing would not lead to dramatic changes of the results, qualitatively speaking.

Gross earnings of the firm are given by the instantaneous revenue function $S = S(K)$. Assume that S is twice continuously differentiable, $S(K) > 0$ for $K > 0$, $S'(K) > 0$, $S''(K) < 0$, $S(0) = 0$. [Function $S(K)$ is defined as revenue after maximization with respect to variable inputs, e.g., labour.]

Let $A = A(I)$ denote the rate of adjustment costs incurred when investing at rate I. Assume that A is twice continuously differentiable and $A(0) = 0$, $A'(I) > 0$ for $I > 0$, and $A''(I) < 0$ for $I < I_0$, $A''(I) > 0$ for $I > I_0$. Hence $A(I)$ is a *concave–convex* function. Define the total cost function, $C(I)$, as $C(I) = I + A(I)$. Note that $C(0) = A(0) = 0$ and that C' and A' attain their minima at $I = I_0$, whereas C/I and A/I attain their minima at $I = I_1 > I_0$.

Earnings, after deduction of depreciation and adjustment costs, can be used to pay out dividends and/or increase equity. Thus

$$S(K) - aK - A(I) = D + \dot{X}. \tag{3}$$

Under the assumption of pure equity financing we can fix the unit value of the capital stock at one unit of money. Using (2) and (3) then implies $K = X$, and we obtain

$$D = S(K) - I - A(I). \tag{4}$$

Substitution of (4) into (1) gives the objective functional

$$\int_0^T \exp(-rt)[S(K) - C(I)]\,\mathrm{d}t + \exp(-rT)K(T). \tag{1a}$$

Lower bounds on dividends and investment are given by

$$D = S(K) - C(I) \geq 0, \tag{5}$$

$$I \geq 0. \tag{6}$$

Inequality (5) states that total (investment) costs must not exceed total current revenue. The constraint (5) has been considered in, for instance, Appelbaum and Harris (1978) and Koskenkylä (1985). [These authors, however, did not include adjustment costs in the objective of the firm.] Negative dividends can be thought of as issues of new equity capital and nonnegativity of dividends could then be interpreted to mean that the firm cannot attract new equity. Inequality (6) is the usual assumption of irreversibility. The implication is that if the firm wishes to reduce its capital stock it can only be done at a speed dictated by the depreciation rate of the capital stock.

The following assumption ensures that the capital stock increases when investment is undertaken at its maximal rate, equalling $C^{-1}(S(K))$ [cf. (5)].

Assumption 1

$$S(K) - aK - A(aK) = S(K) - C(aK) > 0. \tag{7}$$

It has the interpretation that profits (after depreciation and adjustment costs) are strictly positive for every admissible K, when investment is at the replacement level.

The decision problem of the firm is to determine an investment path, $I(t)$, over a fixed and finite planning period $[0, T]$, such that the objective functional in (1a) is maximal, subject to the constraints (2), (5), and (6).

3. Analysis of the investment problem

The investment problem posed in section 2 is analyzed by using optimal control theory. Applying optimal control theory to dynamic economic problems is the subject of, for example, Kamien and Schwartz (1981), Seierstad and Sydsæter (1987), and Feichtinger and Hartl (1986). We have chosen to refer to the latter although most of the theoretical results we shall employ can also be found in the other two references.

Using standard optimal control theory we define the current-value Hamiltonian by $H(I, K, p) = S(K) - paK + pI - C(I)$, where p is a (current-value) adjoint variable. The adjoint equation is

$$\dot{p} = (r + a)p - S'(K), \qquad p(T) = 1. \tag{8}$$

Integration in (8) shows that p is strictly positive for all $t \in [0, T]$, which makes sense when observing that p has the usual interpretation as a shadow price (of a unit) of capital stock at date t.

Disregarding for the moment the upper bound on I, given by $I \leq C^{-1}(S(K))$, an investment policy which maximizes the Hamiltonian for an arbitrary but fixed pair (K, p) is characterized by

$$I = \begin{cases} I(p) & \text{if} \quad p > p^*, \\ 0 \quad \text{or} \quad I_1 & \text{if} \quad p = p^*, \\ 0 & \text{if} \quad p < p^*, \end{cases} \tag{9}$$

where p^* is determined by

$$C'(I_1) = C(I_1)/I_1 = p^*. \tag{10}$$

In (9) $I(p)$ is the solution of $C'(I) = p$, that is, $I(p) = (C')^{-1}(p)$. Thus, the 'smooth' policy $I(p)$ equates marginal investment costs to the shadow price of the capital stock. It holds that $I(p) > I_1$. As in Tobin's q-theory, investment is an increasing function of the shadow price of capital [Tobin (1978), Hayashi (1982)].

The derivation of the policy (9) is omitted since the result appears in Davidson and Harris (1981, eq. (7)). In fig. 1 we have illustrated the argument leading to the policy. The figure suggests that due to the nonconvexity of the cost function an optimal solution may fail to exist if $p = p^*$. This case is important and the issue will be discussed in detail in section 3.1.

To take the constraint $C(I) \leq S(K)$ into account we must modify the investment policy. There are two cases to consider.

Case 1. $I_1 \leq C^{-1}(S(K))$

$$I = \begin{cases} \min\{I(p), C^{-1}(S(K))\} & \text{if} \quad p > p^*, \\ 0 \quad \text{or} \quad I_1 & \text{if} \quad p = p^*, \\ 0 & \text{if} \quad p < p^*. \end{cases} \tag{11a}$$

Case 2. $I_1 > C^{-1}(S(K))$

$$I = \begin{cases} C^{-1}(S(K)) & \text{if} \quad p > S(K)/C^{-1}(S(K)), \\ 0 \quad \text{or} \quad C^{-1}(S(K)) & \text{if} \quad p = S(K)/C^{-1}(S(K)), \\ 0 & \text{if} \quad p < S(K)/C^{-1}(S(K)). \end{cases} \tag{11b}$$

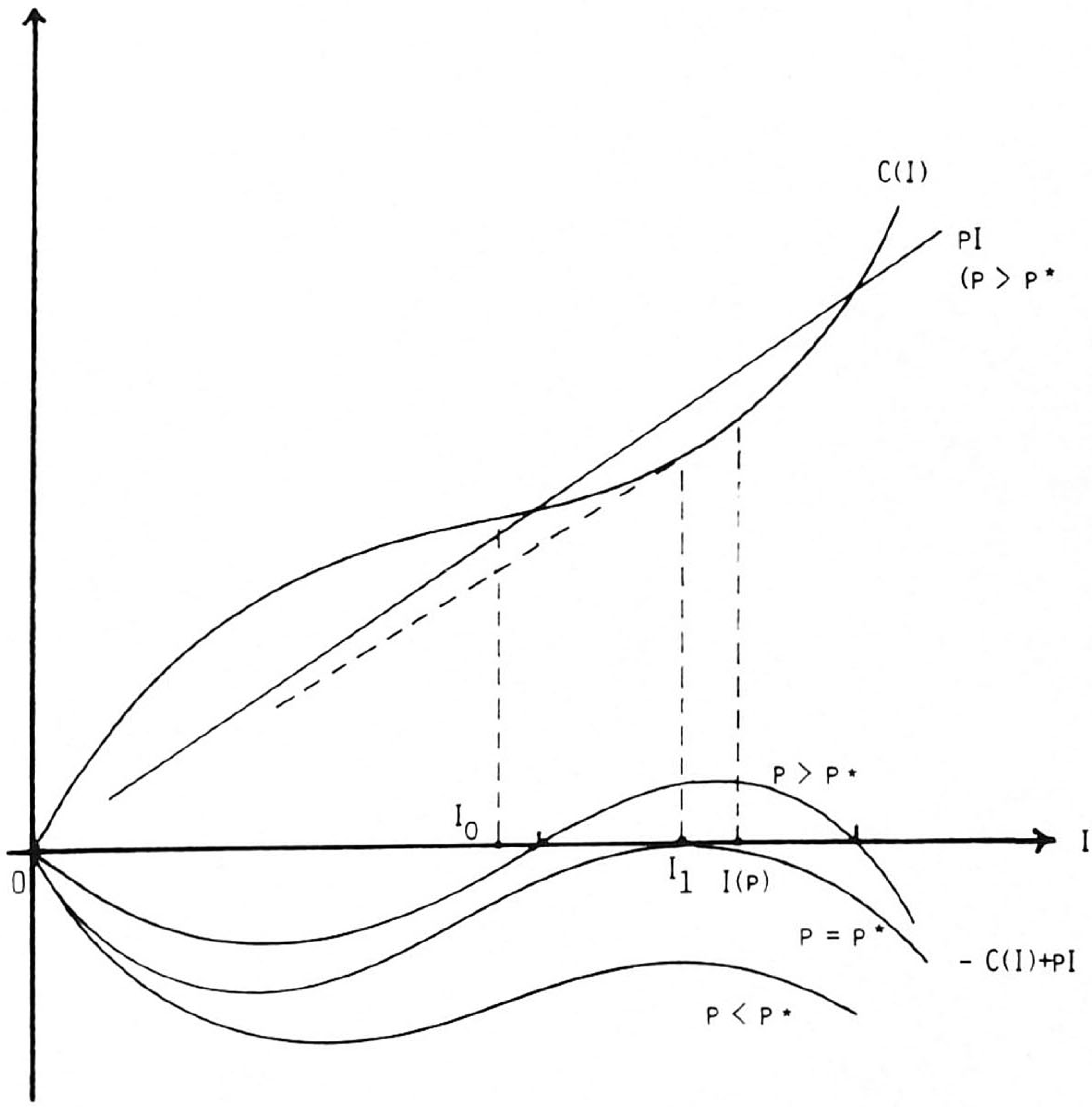

Fig. 1. Maximization of the Hamiltonian.

The derivation of policy (11) from policy (10) is straightforward and follows the same lines of reasoning as in Feichtinger and Hartl (1986, sect. 13.1.4).

The transversality condition in (8) implies that there is always a terminal interval on which the optimal investment rate is zero. This kind of economic behaviour is not implausible and has been noted in a number of optimal control models with a fixed and finite horizon.

We have given a preliminary characterization of an investment policy being a candidate for an optimal solution. The further analysis will proceed as follows. In section 3.1 we analyze a model with a globally concave adjustment cost function; a model with a globally convex adjustment cost function is studied in section 3.2. Then we combine the results of these two analyses to get at a solution to the problem with a concave–convex adjustment cost function (section 3.3). Chattering control will arise under the assumptions of sections 3.1 and 3.3, and section 3.4 is devoted to a discussion of how to avoid such a policy.

3.1. Concave adjustment cost function

A similar model was studied by Kort (1989) but with a fixed upper bound on the investment rate. [Recall that in the paper at hand the term $C^{-1}(S(K))$ provides an endogenous upper bound on the investment rate.]

Let $C(I)$ be a strictly *concave* cost function. Following the standard procedure [Feichtinger and Hartl (1986, p. 414)] we replace function $C(I)$ by a function, $C_1(I, K)$ say, being linear in I such that

$$C_1(I, K) = IS(K)/C^{-1}(S(K)). \tag{12}$$

For any K the function $C_1(I, K)$ coincides with $C(I)$ at $I = 0$ and at $I = C^{-1}(S(K))$ but $C_1(I, K)$ is below $C(I)$ for $0 < I < C^{-1}(S(K))$.

We solve the problem with the linearized cost function (12). To simplify the notation let $S(K)/C^{-1}(S(K)) = g(K)$. The objective functional for this problem becomes

$$\int_0^T \exp(-rt)[S(K) - Ig(K)]\,\mathrm{d}t + \exp(-rT)K(T), \tag{1b}$$

and substituting $I = \dot{K} + aK$ into (1b) yields

$$\int_0^T \exp(-rt)\left[S(K) - \dot{K}g(K) - aKg(K)\right]\mathrm{d}t + \exp(-rT)K(T). \tag{1c}$$

The functions

$$M(K) = S(K) - g(K)aK, \qquad N(K) = -g(K)$$

are continuously differentiable, and we know [see Feichtinger and Hartl (1986, sect. 3.3)] that a singular solution of the linearized problem satisfies $rN(K) + M'(K) = 0$, that is,

$$-rg(K) + S'(K) - ag(K) - aKg'(K) = 0, \tag{13}$$

where

$$\begin{aligned} g'(K) &= \mathrm{d}\{S(K)/C^{-1}(S(K))\}/\mathrm{d}K \\ &= S'(K)/C^{-1}(S(K)) \\ &\quad - S'(K)S(K)/C'(S(K))(C^{-1}(S(K)))^2. \end{aligned} \tag{14}$$

Define the set

$$\Omega(K) = \{I - aK | 0 \le I \le C^{-1}(S(K))\} = [-aK, C^{-1}(S(K)) - aK]$$

and notice that $0 \in \Omega$ because of (7). This implies that the singular solution is sustainable. Inserting (14) into (13) yields

$$S'(K) - (a+r)S(K)/C^{-1}(S(K)) - aKS'(K)/C^{-1}(S(K))$$

$$+ aKS'(K)S(K)/C'(S(K))(C^{-1}(S(K)))^2 = 0. \tag{13a}$$

If no feasible solution(s) exists to (13a) and, in such a case, the left-hand side of (13a) is negative (positive) for all admissible K, it is optimal to invest so as to approach $K = 0$ ($K = \max$) as fast as possible, starting from K_0. From an economic point of view such a solution is obviously an extreme one.

If (13a) has a *unique* solution, K^* say, such that $0 \in \Omega(K^*)$, $K^* > 0$, and the left-hand side of (13a) is *positive* for $K < K^*$, *negative* for $K > K^*$, it is optimal to invest so as to approach K^* as fast as possible, starting from K_0. We already know that $0 \in \Omega(K^*)$. In order to secure the rest of the properties of the solution to (13a), which then guarantee a well-defined 'Most Rapid Approach Path' (MRAP), we impose the following:

Assumption 2

(i) $\mathrm{d}(rN(K) + M'(K))/\mathrm{d}K < 0$ for all $K > 0$,
(ii) $rN(0) + M'(0) > 0$.

The implication of Assumption 2 is the existence of a $K^* > 0$ being the unique solution to (13a). A tedious derivation (the details of which can be obtained from the authors) shows that (i) holds when the discount rate is not 'too large' and functions $S(K)$ and $C(S(K))$ do not grow 'too fast'. Sufficient for the satisfaction of (ii) is that $S'(0)/(a+r) > C'(0)$. In economic terms this means that the marginal profit derived from the first unit of capital stock, corrected for depreciation and discounting, exceeds the amount $C'(0)$.

The optimal policy for the linearized problem is to proceed along an MRAP from K_0 to K^* by using the investment policy

$$I^* = \begin{cases} C^{-1}(S(K)) & \text{if} \quad 0 \le K < K^*, \\ aK^* & \text{if} \quad K = K^*, \\ 0 & \text{if} \quad K > K^*. \end{cases} \tag{15}$$

We suppose that the planning period is sufficiently long such that the unique optimal trajectory associated with I^* in (15) will reach the steady state value K^* at some instant $t < T$.

Recall from (12), (1a), and (1b) the relationship between the problem with concave cost function $C(I)$ and the linearized problem with cost function $C_1(I, K)$, and notice that both problems have the same set of admissible solutions. For any admissible policy $I(t)$ it holds that

$$\begin{aligned} 0 \le J &:= \int_0^T \exp(-rt)[S(K) - C(I)]\,\mathrm{d}t + \exp(-rT)K(T) \\ &\le \int_0^T \exp(-rt)[S(K) - C_1(I, K)]\,\mathrm{d}t + \exp(-rT)K(T) \\ &\le \int_0^T \exp(-rt)[S(K) - C_1(I^*, K)]\,\mathrm{d}t + \exp(-rT)K(T) := J_1 \end{aligned} \tag{16}$$

where the first inequality follows from the constraint $C(I) \le S(K)$ and the second one from the definition of function $C_1(I, K)$. The third inequality follows from (15).

The implication of (16) is that the optimal value of the objective functional of the linearized problem, J_1, imposes an upper bound on the value of the objective functional of the concave problem, J. One of the following two situations occur. If the linearized problem does not admit a singular solution, the solutions of the concave problem and the linear problem are the same [Feichtinger and Hartl (1986, thm. 3.3)]. When the linearized problem has a singular solution, the concave problem does not have an optimal solution [Feichtinger and Hartl (1986, thm. 3.4)].

Since Assumption 2 guarantees the existence of a singular solution to the linearized problem we conclude that the concave problem has no optimal solution. As can be seen from (16) no solution of the concave problem will provide a value of J which actually reaches the upper bound J_1. We know that the value J_1 can be approximated within an arbitrary small neighbourhood by using a *chattering* control. Such a control consists of switching the investment rate rapidly between its lower and upper bounds, in order to keep $K(t)$ as close as possible to the singular solution K^* (on time intervals where K^* is optimal).

The use of chattering control implies that the corresponding trajectory cannot be termed 'optimal' in the usual sense of a payoff-maximizing path; the value of the objective can always be improved under any (well-defined) chattering control by switching a bit more rapidly [Lewis and Schmalensee (1982, p. 101)]. Thus, in the case of a concave adjustment cost function an

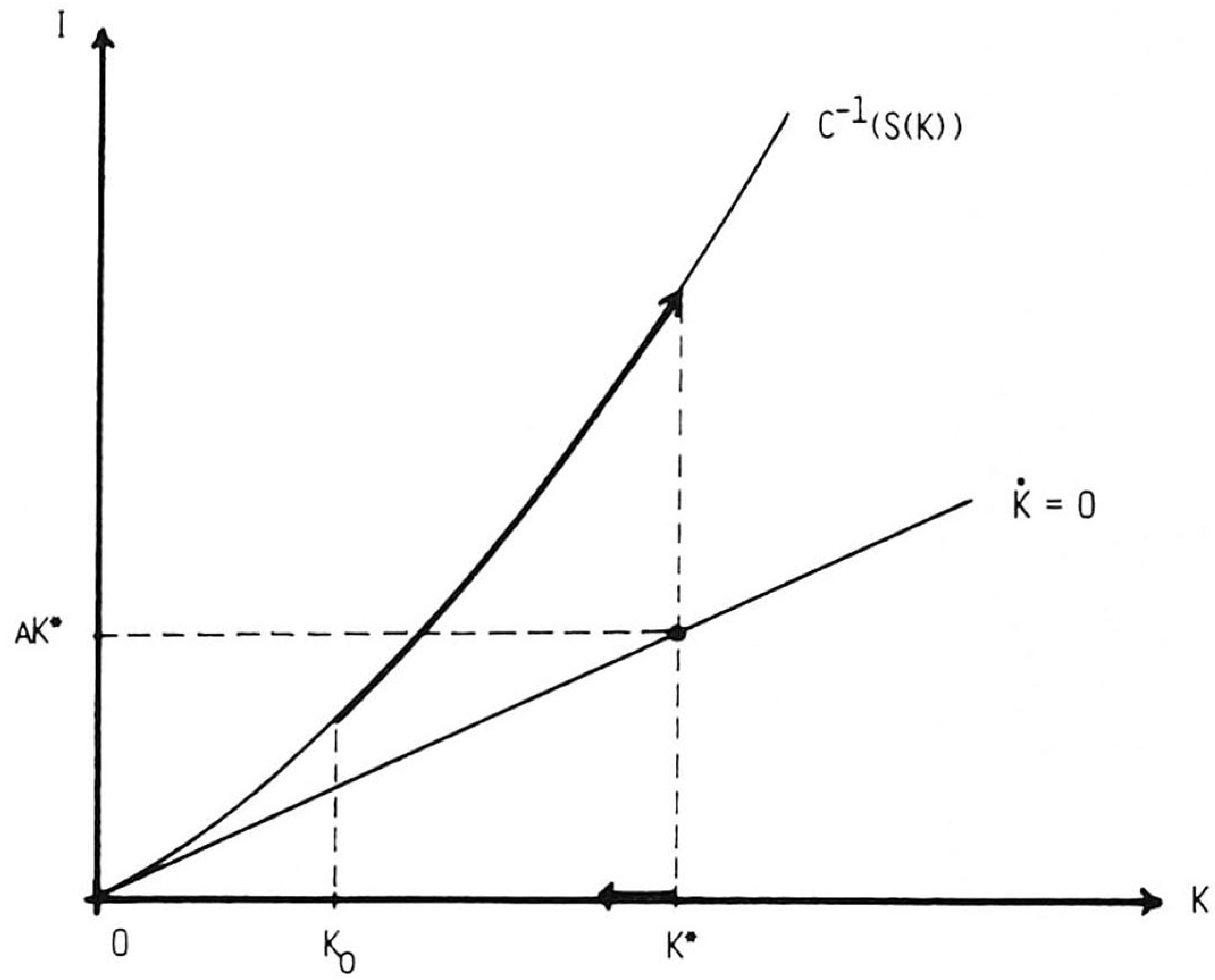

Fig. 2. Optimal solution to the concave adjustment cost problem for the case $K_0 < K^*$.

optimal solution in the usual sense does not exist. The best choice is to apply chattering control and, through intensified chattering, both the optimal trajectory and the optimal value of the objective of the linear problem can be approximated with an arbitrarily small degree of accuracy. The purpose of chattering control is to keep K 'close' to the optimal, but unobtainable K^*.

Using the terminology of Davidson and Harris (1981), a solution containing a phase of chattering control will be denoted as *epsilon-optimal*. The epsilon-optimal solution of the concave adjustment cost problem is depicted in fig. 2. We have chosen to illustrate the plausible case of a firm starting out with a relatively small initial stock of capital, $K_0 < K^*$. If the initial amount of capital goods K_0 is sufficiently low it is optimal to start out with maximal investment, $I = C^{-1}(S(K))$, due to the high marginal earnings [K_0 is relatively low and hence $S'(K)$ is high]. The chattering control is represented by a dot.

We summarize the analysis of the case of a concave adjustment cost function by noting that we have established:

Proposition 1. *With a strictly concave adjustment cost function and for $K_0 < K^*$ [K^* being the unique solution of (13a)], an epsilon-optimal investment policy consists of applying the maximal investment rate, $I = C^{-1}(S(K))$, on an initial interval. This is followed by an intermediate interval where the* *investment rate chatters between $I = 0$ and $I = C^{-1}(S(K))$. On a terminal interval the investment rate is zero.*

3.2. Convex adjustment cost function

This case has been studied extensively and a brief account of the main results will suffice. In this section $C(I)$ is a strictly *convex* cost function.

Remark 2. Now the Hamiltonian as well as function $C^{-1}(S(K))$ are concave in (K, I). Moreover, both the salvage value function and the constraint $I \geq 0$ are linear. Hence the necessary conditions are also sufficient [Feichtinger and Hartl (1986, thm. 7.1)].

We summarize the results for the convex case in the following proposition which is proved in Kort (1989, app. 2).

Proposition 2. For the problem with a strictly convex adjustment cost function the optimal investment policy is characterized by a sequence of the following paths:

$$I = \begin{cases} C^{-1}(S(K)) & (\textit{Path 1}), \\ I(p) = (C')^{-1}(p) & (\textit{Path 2}), \\ 0 & (\textit{Path 3}). \end{cases} \tag{17}$$

Depending on the specific set of model parameters one of the following policy sequences is optimal.

I. Path 1 → Path 2 → Path 3,
II. Path 2 → Path 3,
III. Path 3 → Path 2 → Path 3,
IV. Path 3,

where

$$\text{Sequence} \begin{cases} \text{I} & \text{occurs if} \quad NPVMI > 0 \\ \text{II} & \text{occurs if} \quad NPVMI = 0 \\ \text{III, IV} & \text{occurs if} \quad NPVMI < 0 \end{cases} \quad \text{at} \quad t = 0.$$

NPVMI represents the number 'net present value of marginal investment'. The expressions for *NPVMI* on the various paths can be found in Kort (1989).

A steady state point $(\hat{K}, \hat{I})$ is characterized by

$$S'(\hat{K}) = (r + a)C'(a\hat{K}), \qquad \hat{I} = a\hat{K}, \tag{18}$$

and is a saddlepoint in the K–I plane.

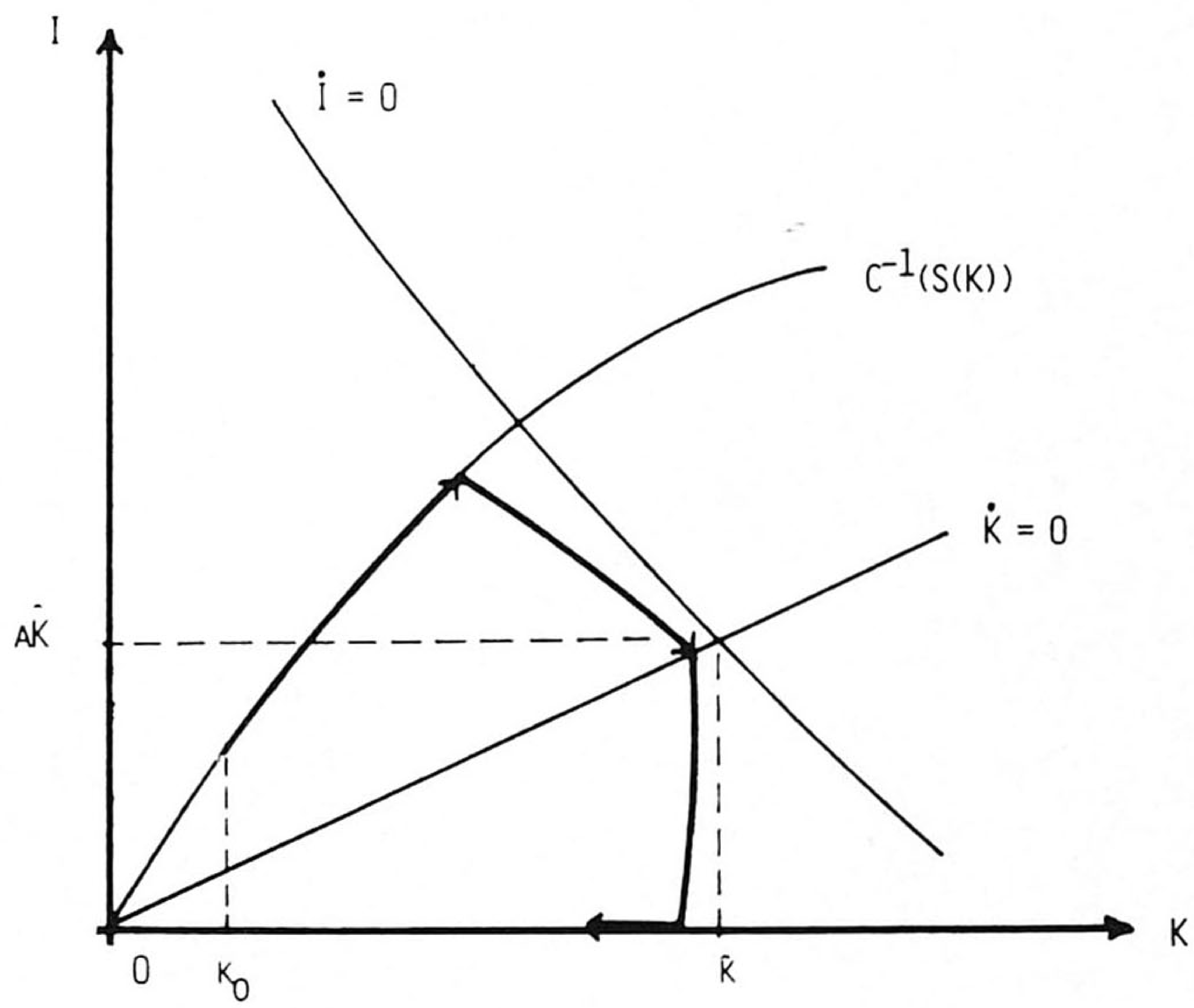

Fig. 3. Optimal solution to the convex adjustment cost problem for the case $K_0 < \hat{K}$.

Trajectory *I* seems to be the most interesting, from an economic point of view. It is depicted in fig. 3. It is optimal to start by investing at the maximal rate $I = C^{-1}(S(K))$ followed by the interior policy $I(p)$ where investment is nonincreasing over time. The final phase of zero investment also occurs here.

3.3. *Concave–convex adjustment cost function*

In order to avoid unnecessary repetition assume in the sequel that K_0 is sufficiently small such that an optimal investment policy always has an *initial phase of maximal investment*, that is, $I = C^{-1}(S(K))$ on an initial interval, $[0, t_1]$ say. During this initial growth phase no dividends are paid out as all revenues are used for investment. The investment rate and the stock of capital are both increasing over time [due to Assumption 1; see also Kort (1989, p. 142)]. At the instant t_1 this growth phase stops since marginal earnings have become too small for a maximal investment policy to be optimal. [Note that marginal earnings, $S'(K)$, decrease when K increases.]

Recall that there is always a *terminal interval*, $[t_2, T]$ say, of *zero investment*. On this final interval contraction occurs: all profits are distributed as dividends and the stock of capital goods decreases.

Remark 3. We use the generic symbols t_1, t_2, and t_3 to indicate instants where the investment policy switches from one type to another. In order not

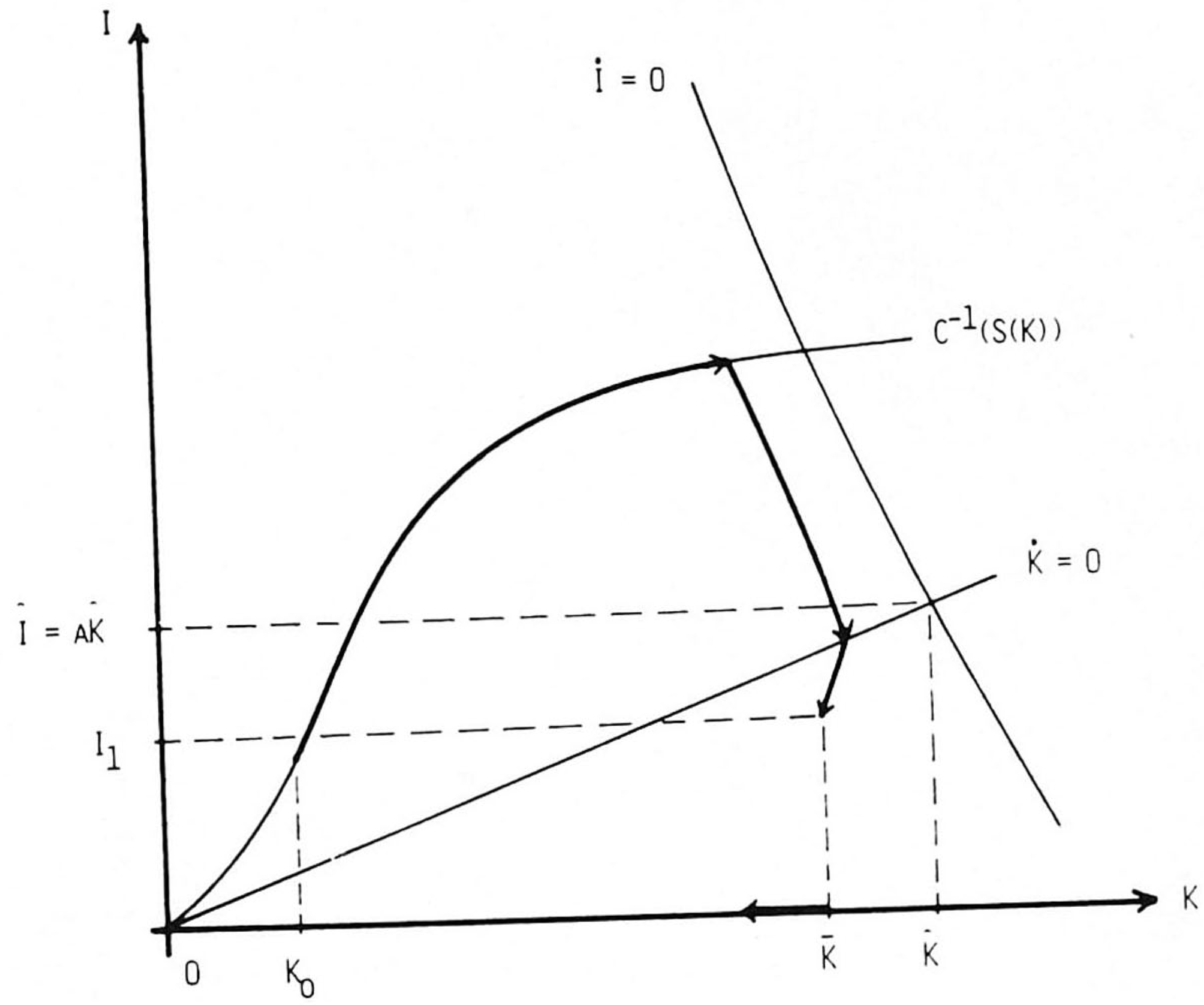

Fig. 4. Optimal solution to the concave–convex adjustment cost problem: Case (i).

to complicate the notation unnecessarily we have chosen to employ the symbols t_1, t_2, and t_3 in all of the three cases to be considered below. However, it is obvious that a particular switching instant will depend on the characteristics of a particular case.

We start out with two preliminary observations. First notice that the phase diagram for the convex case (fig. 3) is valid if $I > I_1$. Second, for $I \leq I_1$ the results for the concave case (fig. 2) will apply with a slight modification. With $\hat{I}$ and $\hat{K}$ given by (18), (7) yields $a\hat{K} < C^{-1}(S(\hat{K}))$. Now, one and only one of the following inequalities is true.

Case (i): $\quad I_1 < \hat{I} < C^{-1}(S(\hat{K}))$,

Case (ii): $\quad \hat{I} < I_1 < C^{-1}(S(\hat{K}))$,

Case (iii): $\quad \hat{I} < C^{-1}(S(\hat{K})) < I_1$.

Case (i). Refer to fig. 4.

Since $I > I_1$ we can use the results of section 3.2 (the convex cost function). We start by stating:

Lemma 1. Let t_2 be the instant at which $I(p)$ becomes equal to I_1. For a sufficiently small $\delta > 0$ it holds that

$$\begin{aligned} I &= I(p) \geq I_1 \quad &&\text{for} \quad t_2 - \delta < t \leq t_2, \\ I &= 0 &&\text{for} \quad t_2 < t < t_2 + \delta. \end{aligned}$$

Proof. Omitted. Can be obtained from the authors.

At the instant t_1 the firm switches from the maximal investment rate to the 'interior' investment policy, $I = I(p)$ [defined in (9)]. During the interval $[t_1, t_2]$ the investment rate is continuously nonincreasing with respect to time; the capital stock first increases but then starts to decrease when the investment rate falls below the replacement level. Since investment is not at its upper bound, a positive amount of dividends is paid out. At time t_2 the investment rate becomes equal to I_1 and then jumps to zero.

For Case (i) we collect our results in:

Proposition 3(i). If I_1 and $\hat{I}$ are such that $I_1 < \hat{I} < C^{-1}(S(\hat{K}))$, an optimal investment policy is given by

$$I = \begin{cases} C^{-1}(S(K)) & \text{for} \quad 0 \leq t < t_1, \\ I(p) & \text{for} \quad t_1 \leq t \leq t_2, \\ 0 & \text{for} \quad t_2 < t \leq T. \end{cases} \tag{19}$$

Remark 4. We have seen that the investment rate drops from $I = I(p) = I_1$ to $I = 0$ 'just after' time t_2. In contrast, in the problem with a strictly convex adjustment cost function (cf. fig. 3) there is an interval, preceding the final phase of zero investment, on which the investment rate is continuously decreasing towards zero. However, if the policy to be employed in Case (i) included such an interval (on which I is lower than I_1), then such an investment policy would be unprofitable because of its higher average cost, $C(I)/I$.

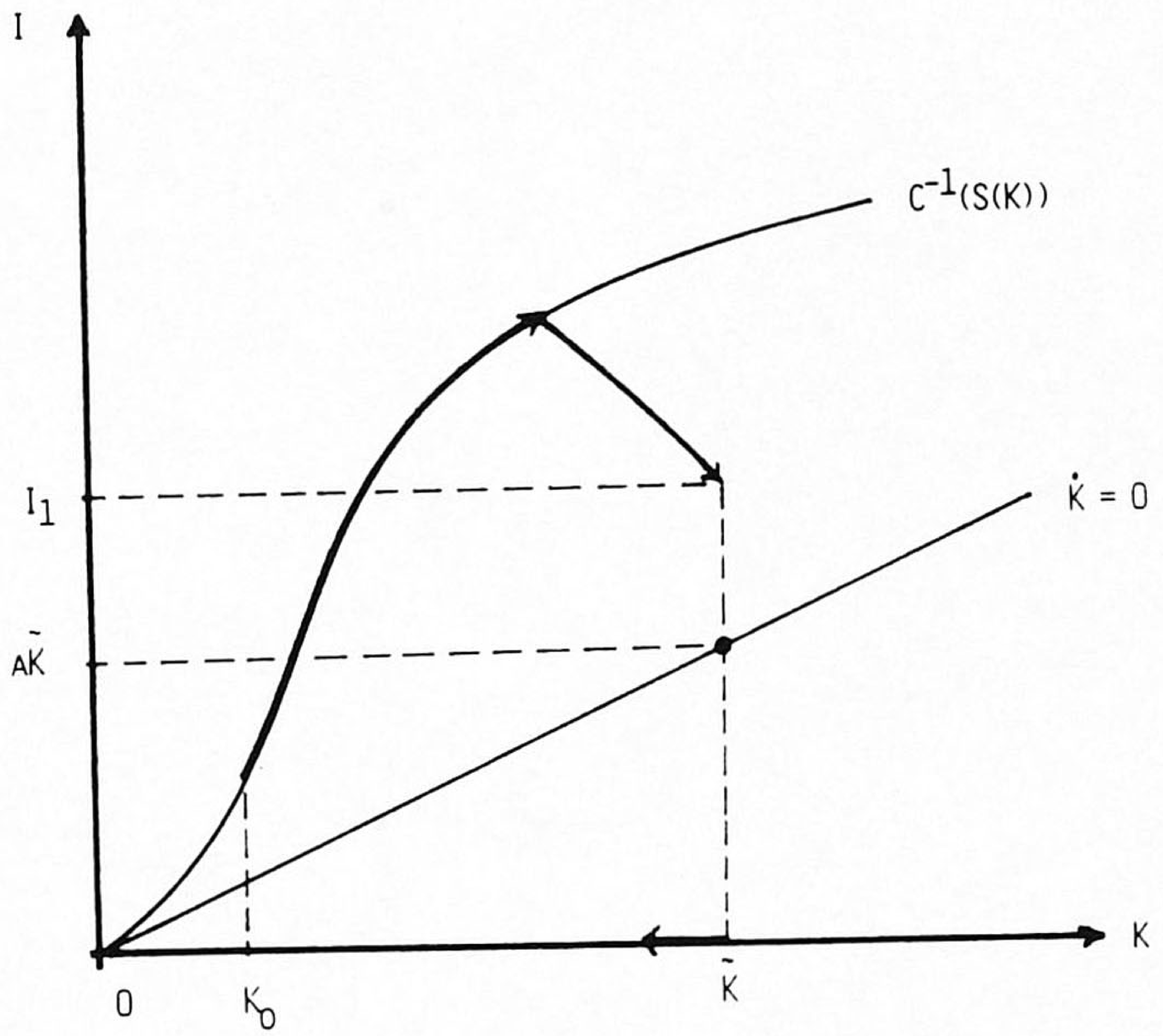

Fig. 5. Optimal solution to the concave–convex adjustment cost problem: Case (ii); the dot represents a chattering control policy.

Case (ii). Refer to fig. 5.

Lemma 2. Define a level of capital, $\tilde{K}$ say, by $S'(\tilde{K}) = (a + r)C'(I_1)$ and let t_3 be an instant of time such that $t_1 < t_3 < t_2$. There exists an interval $[t_1, t_3]$ on which $I = I(p)$ and investment decreases on this interval until $I = I_1$, at which point it holds that $K = \tilde{K}$. The instant of time when this happens is t_3. On the interval $[t_3, t_2]$ investment chatters between 0 and I_1 (in order to keep K as close as possible to $\tilde{K}$).

Proof. Omitted. Can be obtained from the authors.

At the instant t_1 the firm should start to let the investment rate decrease. On the interval $[t_1, t_3]$ investment is kept at its 'interior' level $I = I(p)$ which is characterized by a continuously nonincreasing investment rate [cf. Case (i)]. In contrast to Case (i), the investment rate now reaches (at time $t = t_3$) the line $I = I_1$ *before* falling below the replacement level.

The capital stock increases on the whole interval $[t_1, t_3]$. At time $t = t_3$ the capital stock reaches its equilibrium value $\tilde{K}$ which is characterized by marginal earnings, $S'(\tilde{K})/(a + r)$, being equal to marginal costs, $C'(I_1)$.

For Case (ii) we summarize our results in the following:

Proposition 3(ii). If I_1 and $\hat{I}$ are such that $\hat{I} < I_1 < C^{-1}(S(\hat{K}))$, an epsilon-optimal investment policy is given by

$$I = \begin{cases} C^{-1}(S(K)) & \text{if} \quad 0 \le t < t_1, \\ I(p) & \text{if} \quad t_1 \le t \le t_3, \\ \text{chattering between } 0 \text{ and } I_1 & \text{if} \quad t_3 < t < t_2, \\ 0 & \text{if} \quad t_2 \le t \le T. \end{cases} \tag{20}$$

The difference between Case (ii) and Case (i) lies in the fact that in Case (ii) the 'interior' policy $I(p)$ is not extended to the full length of the intermediate interval $[t_1, t_2]$. A period of chattering investment [on the interval (t_3, t_2)] precedes the final interval of zero investment.

Remark 5. One might argue that the firm should employ an investment policy which maintains the capital stock at the level $\tilde{K}$. Such a replacement policy would, however, be suboptimal since it would induce a higher average investment cost (note that $a\tilde{K} < I_1$). A policy with a lower cost is to keep K as close as possible to $\tilde{K}$ by using a chattering control where the investment rate switches rapidly between 0 and I_1. This is precisely what happens on the interval (t_3, t_2).

Case (iii). Refer to fig. 6.

Some details of this case have already been discussed in section 3.1 (cf. fig. 2). The initial policy of maximal investment is followed by chattering such that I switches as fast as possible between 0 and $C^{-1}(S(K))$ on the interval $[t_1, t_2)$, in order to keep K as close as possible to the singular level K^* [given by (13a)]. As in Case (ii) [cf. Remark 5], a chattering policy is better than replacement investment because the chattering policy carries a lower adjustment cost than replacement investment. We summarize by stating:

Proposition 3(iii). If I_1 and $\hat{I}$ are such that $\hat{I} < C^{-1}(S(\hat{K})) < I_1$, an epsilon-optimal investment policy is characterized by

$$I = \begin{cases} C^{-1}(S(K)) & \text{if} \quad 0 \le t < t_1, \\ \text{chatter between } 0 \text{ and } C^{-1}(S(K)) & \text{if} \quad t_1 \le t < t_2, \\ 0 & \text{if} \quad t_2 \le t \le T. \end{cases} \tag{21}$$

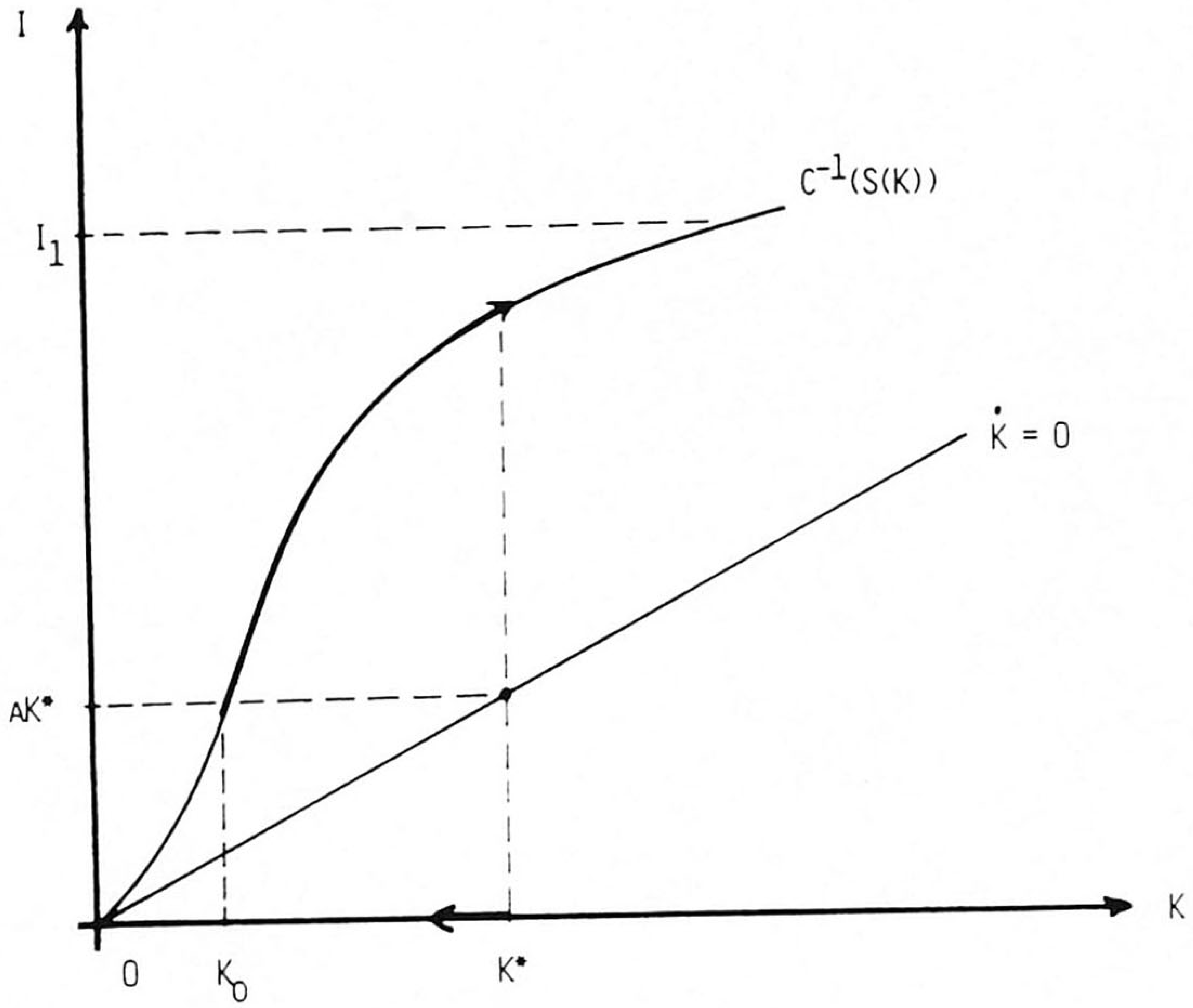

Fig. 6. Optimal solution to the concave–convex adjustment cost problem: Case (iii); the dot represents a chattering control policy.

The solution in Case (iii) has the same structure as the solution of the strictly concave adjustment cost model. This is due to the relatively high value of I_1, below which the average cost function is decreasing.

Comparing Case (iii) with Case (ii) reveals that in the latter the intermediate phase consists of an interval where the investment rate is 'interior' (continuous in time) *and* an interval of chattering control. Thus, the influence of the assumption of a concave–convex adjustment cost function is perhaps most clearly reflected in Case (ii). [Recall that in Case (ii) the value of the critical parameter I_1 is neither particularly high nor low.]

Comparing the three solutions in the concave–convex case with the 'normal' case of a convex cost function (cf. section 3.2), we observe that in the latter the investment rate is continuous with respect to time throughout the interval $[0, T]$. Of the three solutions of the concave–convex adjustment cost model, Case (i) is the one which comes closest to having this property; the policy in Case (i) has only one discontinuity (at $t = t_2$).

3.4. Exclusion of chattering policies

A point of critique which can be raised against the optimal investment policies in sections 3.1 and 3.3 is the occurrence of chattering controls. A chattering policy, alternating between zero and 'high' levels of investment, can be interpreted to imply in practice, the faster the switching the better. Although this interpretation implies that a 'pulsing' policy with the highest possible pulsing frequency may be the most profitable, the chattering itself is not an implementable policy. The use of chattering control is a mathematical convenience rather than a reflection of observable real-world behaviour. However, avoiding the occurrence of chattering policies requires a change of the model assumptions.

In what follows we show that an optimal policy involving chattering can be prevented by introducing start-up costs or by imposing adjustment costs on the rate of change of the investment rate. Other alternatives would be:

(i) to include a possibility of impulsive investment, that is, investment takes place in 'lumps' [Kort (1989)],
(ii) a convex–concave revenue function in association with a strictly convex adjustment cost function [Davidson and Harris (1981), Dechert (1984)],
(iii) a convex–concave revenue function with concave–convex adjustment costs (this case may be an interesting topic for future research).

3.4.1. Start-up costs

One possibility is to incorporate a start-up cost which is incurred every time the firm raises the investment rate from zero to a positive level [see, for example, Davidson and Harris (1981, sect. 4)]. Incidentally, such a cost plays the same role as the 're-entry' cost in the theory of optimal extraction of renewable resources [Lewis and Schmalensee (1982)] or a 'pulsing cost' in models of optimal advertising [Hahn and Hyun (1991)]. Other examples can be found in production R&D planning [Feichtinger and Sorger (1986, 1988)]. From studies like these it is known that the introduction of set-up costs prevents the occurrence of chattering.

The problem is to choose an investment rate so as to maximize

$$\int_0^T \exp(-rt)[S(K) - C(I) - \delta(I)F]\,\mathrm{d}t + \exp(-rT)K(T) - \sum_{t\in\Theta} \exp(-rt)R, \tag{21}$$

subject to (2) and (6). In (21) F represents a flow-fixed cost incurred whenever I is positive; $\delta(I)=1$ for $I>0$, $\delta(I)=0$ otherwise. The term $C(I)+\delta(I)F$ denotes the total cost flow. R represents the start-up cost (fixed and of stock dimension) incurred at all times t where $I(t)$ goes from zero to a positive level. The set Θ is the collection of all such instants. A constraint analogous to (5) is the following:

$$S(K) - C(I) - \delta(I)F - \beta(t)R \geq 0, \tag{22}$$

where $\beta(t) = 1$ if $t \in \Theta$, $\beta(t) = 0$ otherwise.

The introduction of the set-up cost, R, surely penalizes the use of chattering investment to offset the nonconvexity of the cost function. On the other hand, the flow-fixed cost, F, induces the firm to use a zero investment policy for longer periods of time. Davidson and Harris (1981) studied the problem (21) with T infinite but without the constraint (22).

Consider the following candidate policies:

(a) *Chattering investment policy*

(b) *Periodic (cycling, pulsing) investment policy*

where chattering means that I switches rapidly between $I = I_L \geq 0$ and $I = I_U > I_L$ in order to keep the trajectory $K(t)$ as close as possible to a desired level. [Lewis and Schmalensee (1982) used the term 'vibrating' for the limiting case where switching takes place infinitely fast.] A (long-run) periodic policy is defined as follows. There exists an instant $\tau \geq 0$ and a constant $\pi > 0$ such that for all $t \geq \tau$ it holds that $K(t) = K(t + \pi)$ and $I(t) = I(t + \pi)$. [The caveat 'long-run' refers to the fact that the periodic policy may first be applied after the passing of an initial interval, i.e., $\tau > 0$.]

Davidson and Harris (1981) showed that nonconvexity of the adjustment cost function is a necessary condition for the start-up costs to have an influence on the optimal steady state investment policy. It is, however, not a sufficient condition. If the adjustment cost function is nonconvex and $F = 0$, then $C(I) + \delta(I)F \to 0$ for $I \to 0$. Thus $C(I_L) + \delta(I_L)F$ is arbitrarily small when investment chatters between an arbitrarily small, but *positive* I_L, and I_U. This is called 'positive chattering' and it prevents the payment of start-up costs because the set Θ is empty. The upshot is that the start-up cost, R, has a (long-run) effect only if the cost function includes the flow-fixed cost, that is, if $F > 0$.

The firm may choose a periodic policy with zero investment on some intervals, thus escaping the flow fixed cost, F, but at the expense of having the set-up cost R at the end of such intervals. Conversely, the firm may avoid the cost R by using positive chattering but then it continuously incurs the cost F. Davidson and Harris (1981) show that if, roughly stated, starting up is 'cheap' or the flow-fixed cost is 'high', a periodic policy is optimal. On the other hand, if starting up is 'expensive' or the flow-fixed cost is 'low', a positive chattering policy is optimal. [Similar results are reported in Feichtinger and Sorger (1986, 1988).]

These results make economic sense, and we conjecture that they carry over to this paper's problem; the assumptions of a finite planning horizon and an endogenous upper bound on the investment rate [cf. (22)] should not change qualitatively the results obtained by Davidson and Harris (1981) and Feichtinger and Sorger (1986, 1988).

3.4.2. Two kinds of adjustment costs

A second way to avoid chattering is to penalize changes in the investment rate by making such changes costly. Hence, adjustment costs are incurred not only for changing the capital stock but also for changing the (gross) investment rate. [This approach was also used by, e.g., Feichtinger and Sorger (1986, sect. 3).]

Let the cost function depend on the investment rate, I, as well as the rate of change of investment, $\dot{I}$. Total costs equal $B(I, \dot{I})$ and we suppose that function B is separable, that is,

$$B(I, \dot{I}) = C(I) + G(\dot{I}),$$

where C is concave–convex and G is strictly convex. Assume that $G(0) = 0$ and, for the sake of illustration, let G be quadratic,

$$G(I) = \dot{I}^2/2.$$

Introduce I as a second state variable by the state equation

$$\dot{I} = u,$$

such that u now acts as the control variable. Notice that u is unconstrained, u must be piecewise continuous, and I is *continuous*. The optimization problem is as follows. Choose a time path for the control variable u so as to maximize the objective functional

$$\int_0^T \exp(-rt)\left[S(K) - C(I) - u^2/2\right] \mathrm{d}t + \exp(-rT)K(T), \qquad (23a)$$

subject to the dynamic equations

$$\dot{K} = I - aK, \qquad K(0) = K_0 > 0, \qquad (23b)$$

$$\dot{I} = u, \qquad I(0) = I_0 \geq 0, \qquad (23c)$$

and the constraints

$$I \geq 0, \tag{23d}$$

$$S(K) - C(I) - u^2/2 \geq 0. \tag{23e}$$

Define the (current-value) Hamiltonian

$$H = \{S(K) - C(I) - u^2/2\} + m_1(I - aK) + m_2 u,$$

the Lagrangian

$$L = H + \mu\left[S(K) - C(I) - u^2/2\right] + \upsilon I,$$

and the set

$$\Omega(K, I) = \{u \in R \mid S(K) - C(I) - u^2/2 \geq 0\}.$$

The necessary optimality conditions are as follows:

$$u = \underset{u \in \Omega}{\operatorname{argmax}}\, H, \tag{24a}$$

$$L_u = 0 \quad \rightarrow \quad u = m_2/(1 + \mu), \tag{24b}$$

$$\dot{m}_1 = (r + a)m_1 - (1 + \mu)S'(K), \qquad m_1(T) = 1, \tag{24c}$$

$$\dot{m}_2 = rm_2 - m_1 - \upsilon + (1 + \mu)C'(I), \quad m_2(T) = \Gamma, \tag{24d}$$

$$\mu \geq 0, \quad \mu\left[S(K) - C(I) - u^2/2\right] = 0, \tag{24e}$$

$$\upsilon \geq 0, \quad \upsilon I = 0, \tag{24f}$$

$$\Gamma \geq 0, \quad \Gamma I(T) = 0, \tag{24g}$$

where m_i $(i = 1, 2)$ are piecewise continuously differentiable costate variables and μ, υ are piecewise continuous multiplier functions. Γ is a constant multiplier. [The curly bracket in H should be multiplied by a nonnegative constant. It is easy to prove that this constant is positive, and we can put it equal to one.] The costate variable m_1 plays the same role as p in (8).

Consider the maximization of H subject to $u \in \Omega$. The conditions (24b), (24e) are the Kuhn–Tucker conditions for the optimization problem (24a) with (K, I, m_1, m_2) arbitrary but fixed. The Hamiltonian is concave in u, and the function $-S(K) + C(I) + u^2/2$ is convex in u. Hence the Kuhn–Tucker conditions are sufficient as well. Maximization of H yields a unique solution because $H_{uu} < 0$. It follows [Feichtinger and Hartl (1986, corol. 6.2)] that u is continuous, in particular also at points of entry/exit from the state constraint $I \geq 0$. This constraint is a first-order state constraint, and the constraint qualification is satisfied since the matrix

$$\begin{pmatrix} -u & S(K) - C(I) - u^2/2 & 0 \\ u & 0 & I \end{pmatrix}$$

has full (row) rank unless both constraints (23d), (23e) are binding. The latter cannot happen, however, since it would imply $K = 0$ which contradicts (23b), (23d). It follows [Feichtinger and Hartl (1986, corol. 6.3)] that the costate m_2 is continuous. Moreover, when m_2 is continuous it holds that $m_2(T) = \Gamma = 0$. The costate m_1 is continuous since the state constraint $I \geq 0$ contains neither K nor t.

Using (24b) we integrate (24d) to obtain

$$m_2 = \int_t^T \exp\{r(t-s)\}\left[m_1(s) + v(s) - (1 + \mu(s))C'(I(s))\right] \mathrm{d}s. \tag{25}$$

The optimal control u is unique and continuous. The implication is that the time derivative of the investment rate is also continuous which *precludes the investment rate from chattering*.

The complementary slackness conditions yield three types of policies. [A policy where $\mu > 0$ and $v > 0$ is infeasible. On such a path it must hold that $I = u = S(K) = 0$ but, as already noted, the latter equality only holds if $K = 0$, which is excluded by (23b), (23d).]

A: $I > 0,\ \mu > 0 \rightarrow v = 0,\ S(K) - C(I) - u^2/2 = 0$

No dividends are distributed since all revenue is used for paying investment and adjustment costs. The control u can be found in feedback form, $u = u(K, I)$ by solving the equation $S(K) - C(I) - u^2/2 = 0$.

Consider a short interval during which revenue $S(K)$ can be used for paying the cost $C(I)$ and/or $G(u) = u^2/2$. Notice that the former maximal investment rate, $I = C^{-1}(S(K))$, can be sustained only if $G(u) = 0$, implying u equal to zero, i.e., investment is not changing during the interval. Conversely, a rapid change in the investment rate can be undertaken by applying a high value of u [subject to the constraint (23e)]. In such a case the high cost $G(u)$ reduces the amount available to pay the cost $C(I)$. Hence, only 'small' amounts of investment, I, can be carried out.

Thus, on the constrained path under consideration the firm faces a trade-off. If the firm wishes to decrease the cost $G(u)$ in order to increase the amount available for paying the cost $C(I)$ of a high level of investment, only small changes of the investment rate can be made, i.e., the investment rate must be kept relatively constant (but at a high level). If the firm wishes to increase the amount available to pay the cost $G(u)$ in order to make a rapid change of the investment rate, the cost $C(I)$ must be relatively low, and hence investment must be kept at a low level.

B: $I > 0,\ S(K) - C(I) - G(u) > 0 \rightarrow v = \mu = 0$

On this unconstrained path a positive amount of dividends is paid out. The control variable u equals m_2, where

$$m_2 = \int_t^T \exp\{r(t-s)\}\left[m_1(s) - C'(I(s))\right] \mathrm{d}s. \tag{26}$$

The solution (26) assumes that path B qualifies as a final path. This is indeed true as will be demonstrated below. The costate m_1 is given as the solution of (24c) for $\mu = 0$. Note that m_1 is positive for all t.

Eq. (26) shows that the value of u at any instant t depends on the future relationship between the shadow price m_1 of the capital stock and the marginal investment cost. On the interval $[t, T]$ it holds that I is positive (but not maximal) and

$$m_1 \gtreqless C'(I) \rightarrow \dot{I} \gtreqless 0. \tag{27}$$

The results in (27) are easily interpreted and make economic sense.

C: $v > 0,\ S(K) - C(I) - G(u) > 0 \rightarrow I = 0,\ \mu = 0$

It holds that $I = u = 0$ and $C(I) = G(u) = 0$. This is the phase of zero investment encountered earlier: the capital stock decreases exponentially and all revenue is distributed as dividends.

It can be proven that policy A cannot precede policy C, and vice versa. Moreover, it can be proven that policy A is not a feasible final policy. (The proofs are technical and omitted.) Hence the policy string ends with a period of zero investment (that is, policy C, preceded by the interior policy B), *or* the string ends with policy B where investment is positive. Thus, when chattering is excluded there is also a possibility of a terminal interval with nonzero investment rate.

Finally we characterize the optimal u-policy. The constraint (23e) is

$$u \leq \pm\sqrt{2}\,[S(K) - C(I)] := \begin{cases} u^0 > 0, \\ u_0 < 0. \end{cases} \tag{28}$$

We have $H_u = m_2 - u$ and an optimal u^* is characterized by

$$\begin{aligned} u_0 < m_2 < u^0 \quad &\rightarrow \quad u^* = m_2, \\ m_2 \leq u_0 \quad &\rightarrow \quad u^* = u_0, \\ m_2 \geq u^0 \quad &\rightarrow \quad u^* = u^0. \end{aligned}$$

It is easy to prove that a final interval with decreasing but *positive* investment rate now can occur in an optimal solution. When $I > 0$ on a final interval it holds that $\nu = \mu = 0$ because path A is not a feasible final path. Moreover, $m_2(T) = 0$ which implies $u(T) = 0$. Eq. (25) shows that $m_2(T) = C'(I(T)) - 1 > 0$. By continuity, and for a sufficiently small $\tau = \text{const.} > 0$, it must be true that $m_2(T - \tau) > 1$. Hence there is a final interval on which u is negative, that is, I decreases on that interval. [The fact that u is negative follows from (28), and from the observation that infeasibility of path A as a final path implies that $S(K) - C(I) > 0$ must hold.]

Remark 6. If we assume an infinite planning horizon the steady state obviously has $I > 0$ and $u = 0$. Recall that in the case of a finite horizon this situation could not occur on the terminal interval. This seems to suggest that a study of the finite planning horizon case yields some scenarios that are not encountered in an infinite horizon set-up.

4. Concluding remarks

In this paper a dynamic investment model of a firm has been analyzed. The key features were the *concave–convex adjustment cost function*, a *finite planning horizon*, and an *endogenous upper bounded rate of investment*. Depending on the parameter values and the specific functional forms of the earnings and cost functions, three different types of optimal solutions emerged. These solutions were studied in the case of a small initial stock of capital goods.

A common feature of all solutions was an initial growth phase with maximal investment and a terminal contraction phase with zero investment. The latter is driven by the assumption of a finite planning horizon. However, on an intermediate interval the solutions were different. In the first case the investment rate decreased continuously until reaching the value at which the average adjustment costs are minimal. In the second case such a phase also appeared but was followed by a chattering investment policy where the investment rate switched rapidly between zero and the value at which average adjustment costs are minimal. In the third case the intermediate interval only contained a chattering policy where the investment rate switches rapidly between zero and its upper bound.

The phases of initial maximal growth and final contraction did not appear (at least not formally) in Davidson and Harris (1981) who assumed an infinite planning period and no upper bound on the rate of investment. Moreover, the policy to be followed during the intermediate phase has, in the present paper, a richer structure compared to that in Davidson and Harris who obtained an investment policy with chattering between zero and an unspecified value (being greater than or equal to I_1). Introduction of an upper bound on investment made it possible for us to give a precise characterization of the levels of investment to be employed in the chattering policies [Cases (ii) and (iii)]. Due to their assumptions, Davidson and Harris did not need to distinguish between the situations treated in our Cases (ii) and (iii).

As to future research, a possible avenue could be to extend the horizon in the model of section 3.4.2 to infinity and study the limiting behaviour in terms of saddlepoint stability or limit cycles. However, the conditions for the occurrence of limit cycles are not easily interpretable, involving third- and fourth-order derivatives of, for instance, the cost function [see, e.g., Feichtinger and Sorger (1986)].

References

Appelbaum, E. and R. Harris, 1978, Optimal capital policy with bounded investment plans, International Economic Review 19, 103–114.

Brechling, F., 1975, Investment and employment decisions (Manchester University Press, Manchester).

Davidson, R. and R. Harris, 1981, Non-convexities in continuous-time investment theory, Review of Economic Studies 48, 235–253.

Dechert, W.D., 1984, Has the Averch–Johnson effect been theoretically justified?, Journal of Economic Dynamics and Control 8, 1–17.

Eisner, R. and R.H. Strotz, 1963, Determinants of business investment, in: Commission on money and credit, Impacts of monetary policy (Prentice-Hall, Englewood Cliffs, NJ).

El-Hodiri, M. and A. Takayama, 1981, Dynamic behavior of the firm with adjustment costs, under regulatory constraint, Journal of Economic Dynamics and Control 3, 29–42.

Eswaran, M., T. Lewis, and T. Heaps, 1983, On the nonexistence of market equilibria in exhaustible resource markets with decreasing costs, Journal of Political Economy 91, 154–167.

Feichtinger, G. and R.F. Hartl, 1986, Optimale Kontrolle ökonomischer Prozesse (W. de Gruyter, Berlin).

Feichtinger, G. and G. Sorger, 1986, Optimal oscillations in control models: How can constant demand lead to cyclical production?, Operations Research Letters 5, 277–281.

Feichtinger, G. and G. Sorger, 1988, Periodic research and development, in: G. Feichtinger, ed., Optimal control theory and economic analysis 3 (North-Holland, Amsterdam) 121–141.

Gould, J.P., 1968, Adjustment costs in the theory of investment of the firm, Review of Economic Studies 35, 47–56.

Hahn, M. and J.S. Hyun, 1991, Advertising cost interactions and the optimality of pulsing, Management Science 37, 157–169.

Hayashi, F., 1982, Tobin's marginal q and average q: A neoclassical interpretation, Econometrica 50, 213–224.
Jorgenson, D.W., 1963, Capital theory and investment behavior, American Economic Review 52, 247–259.
Kamien, M.I. and N.L. Schwartz, 1981, Dynamic optimization: The calculus of variations and optimal control in economics and management (North-Holland, Amsterdam).
Kort, P.M., 1989, Optimal dynamic investment policies of a value maximizing firm (Springer-Verlag, Berlin).
Koskenkylä, H., 1985, Investment behaviour and market imperfection with an application to the Finnish corporate sector, Series B:38 (Bank of Finland Publications, Helsinki).
Lewis, T.R. and R. Schmalensee, 1982, Optimal use of renewable resources with nonconvexities in production, in: L.J. Mirman and D.F. Spulber, eds., Essays in the economics of renewable resources (North-Holland, Amsterdam) 95–111.
Lucas, R., 1967, Adjustment costs and the theory of supply, Journal of Political Economy 75, 321–334.
Mortensen, D., 1973, Generalized costs of adjustment and dynamic factor demand theory, Econometrica 41, 657–665.
Nickell, S., 1975, A closer look at replacement investment, Journal of Economic Theory 10, 54–88.
Nickell, S., 1978, The investment decisions of firms (J. Nisbet, Welwyn).
Rothschild, M., 1971, On the cost of adjustment, Quarterly Journal of Economics 85, 605–622.
Seierstad, A. and K. Sydsæter, 1987, Optimal control theory with economic applications (North-Holland, Amsterdam).
Sinn, H.-W., 1987, Capital income taxation and resource allocation (North-Holland, Amsterdam).
Söderström, H.T., 1976, Production and investment under costs of adjustment: A survey, Zeitschrift für Nationalökonomie 36, 369–388.
Tobin, J., 1978, Monetary policies and the economy: The transmission mechanism, Southern Economic Journal 44, 421–431.
Treadway, A.B., 1970, Adjustment costs and variable inputs in the theory of the competitive firm, Journal of Economic Theory 2, 329–347.